

Role of control constraints in quantum optimal control

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Have you stopped drinking brandy in the mornings? Answer: Yes or No?

Astrid Lindgren

Karlsson-on-the-Roof is Sneaking Around Again

The problems of optimizing the value of an arbitrary observable of the two-level system at both a fixed time and the shortest possible time is theoretically explored. Complete identification and classification along with comprehensive analysis of globally optimal control policies and traps (i.e. policies which are locally but not globally optimal) is presented. The central question addressed is whether the control landscape remains trap-free if control constraints of the inequality type are imposed. The answer is astonishingly controversial, namely, although formally it is always negative, in practice it is positive provided that the control time is fixed and chosen long enough.

PACS numbers: 03.65.-w, 02.30.Yy, 03.67.Ac, 37.10.Jk

I. INTRODUCTION

Coherent control of the two level system is crucial for qubit design. The two-level Landau-Zener system is probably the most fundamental qubit model with the single control parameter u . Its master equation reads:

$$\rho(\tau) = U_{\tau,0}(u)\rho(0)U_{\tau,0}^\dagger(u), \quad (1)$$

with the unitary transformation $U_{\tau'',\tau'}(u)$ defined as $U_{\tau'',\tau'}(u) = \overrightarrow{\exp}(-i \int_{\tau=\tau'}^{\tau''} (\hat{\sigma}_x + u(\tau)\hat{\sigma}_z) d\tau)$. Here ρ is the system's density matrix, σ_x and σ_z are Pauli matrices, τ is a dimensionless time $\tau = \alpha t$, and the control parameter is usually proportional to the interaction strength with an external controlled electric or magnetic field ($u = \beta \mathcal{E}$ or $u = \beta \mathcal{B}$). Depending on the physical meaning of the scaling factors α and β , Eq. (1) can represent the wide variety of modern experiments on magnetic or optical control of quantum dots [1], vacancy centers in crystals [2], spin states of atoms and molecules [3], Bose-Einstein condensates [4, 5], superconducting circuits [6] etc.

We consider the following optimal control problem:

$$J = \text{Tr}[\rho(T)\hat{O}] \rightarrow \max \quad (2)$$

$$-u_{\max} \leq u \leq u_{\max}, \quad (3)$$

$$T < T_{\max}, \quad (4)$$

where max is taken with respect to the program (or control policy) $\tilde{u}(\tau)$, and possibly also the final time T . This task well represents the initial preparation of the qubit in the given initial pure state corresponding to the largest eigenvalue of the observable \hat{O} .

The key question of our study is the extent to which the restrictions (3), (4) complicate finding the policy $\tilde{u}^{\text{opt}}(\tau)$ which maximizes $J[u]$ using the local search methods. The applied value of this question is justified both by technical limitations and also the breakdown of the two-level approximation in strong fields. In addition, the bound (4) is motivated by the fatal losses of fidelity of quantum gates due to uncontrollable decoherence at long times.

The major obstacles in searching for $\tilde{u}^{\text{opt}}(\tau)$ are “traps” and “saddle points”. These are the special policies \tilde{u} such that $J[\tilde{u}(\tau)] < J^{\text{opt}}$ and $J[\tilde{u} + \delta u] < J[\tilde{u}]$ (trap) or $J[\tilde{u} + \delta u] = J[\tilde{u}]$ (saddle point) for any infinitesimal variation $\delta u(\tau)$ consistent with (3). Presence of saddle points slows down the convergence of any local search strategy whereas reaching the trap leads to its failure. Traps and saddle points are controversial matters for optimal quantum control (OQC) theory. Despite substantial experimental evidence [10] supported by theoretical arguments of trap-free “quantum landscapes” $J[\tilde{u}(\tau)]$ [8, 9, 11, 12], it is also easy to provide simple counterexamples [13–15].

The Landau-Zener system is quite special from this perspective since is the only system for which absence of traps in the unconstrained case (i.e. when $u_{\max} = \infty$ in (3)) was formally proven [18, 19]. Moreover, its complete controllability for any finite value of u_{\max} (provided that T_{\max} is chosen sufficiently long) was also justified [20–22]. Thus, this system provides opportunity to evaluate the effect of constraints (4) and (3) on the landscape complexity in the most pristine form. The existing data portend that this effect should be nontrivial. For example, the unconstrained time-optimal policies $\tilde{u}(\tau)$ are shown to be $\tilde{u}(\tau) = c'\delta(\tau) + c''\delta(\tau - T)$ where c' and c'' are constants and $\delta(\tau)$ is the Dirac delta function [24]. Such solutions are evidently inconsistent with any constraints of form (3).

An additional feature of the Landau-Zener system is its simplicity, which allows us to infer analytically the topol-

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ogy of $J[u]$. We hope that this exceptional opportunity will provide some clue on the expected outcomes in more complex cases, which can be handled only numerically.

It is worth mentioning that the restrictions (3) are critical in the foundation of modern theory of optimal control since the corresponding problems can not be solved in the framework of classical calculus of variations and require special methods, such as the Pontryagin's maximum principle (PMP) [16, 17]. For completeness of the presentation, we provide in Sec. II the brief overview of PMP and the known results of the first-order analysis of controlled Landau-Zener system in the PMP framework. In particular, we clarify why the unconstrained problem (2) is trap-free, and introduce the primary classification of the stationary points (i.e. the locally and globally optimal solutions, traps and saddle points) by showing that all of them in the case of time-optimal control as well as traps and saddle points in the case of fixed time control are represented by piecewise-constant controls $\tilde{u}(\tau)$ which can take only 3 values: 0 and $\pm u_{\max}$.

The rest of the paper is organized as follows. In Sec. III we derive the comprehensive set of criteria which allow to outline the landscape profile and distinguish between various types of its stationary points. The obtained criteria substantially extend, generalize and/or specialize a number of known results [23–26] obtained for related problems using the index theory [27] methods, optimal syntheses on 2-D manifolds [28], etc. In this work we propose the technique of “sliding” variations which allows to reduce the high-order analysis to methodologically simple and intuitively appealing geometrical arguments.

In subsequent sections IV and V we apply these criteria to identify and classify the traps and saddle points for the cases of time-optimal and time-fixed control, correspondingly. A brief summary of the obtained results and the general conclusions which follow from this analysis are given in the final section VI.

II. REGULAR AND SINGULAR OPTIMAL POLICIES

In this section we review the first-order analysis of problem (1) with constraint (3) in the PMP framework. For additional details one can refer to extensive literature (e.g. [17], pp.280-286,[23]). PMP provides the necessary criterion of local optimality of control $u(\tau)$ in terms of the Hamilton-type Pontryagin function $K(\rho(\tau), \hat{O}(\tau), u(\tau))$:

$$\tilde{\mathbf{u}}(\tau) = \arg \max_{u(\tau)} K(\tilde{\rho}(\tau), \tilde{\hat{O}}(\tau), u(\tau)). \quad (5)$$

The processes satisfying the PMP are called stationary points, or extremals, and will be denoted hereafter with the \sim marks: $\{\tilde{u}(\tau), \tilde{\rho}(\tau), \tilde{\hat{O}}(\tau)\}$.

The Pontryagin function among the state variables ρ and controls u depends also on the so called co-state of adjoint variables $\hat{O}(\tau)$, which are subject to the special

evolution equation and boundary transversality conditions. In the case of the control problem 1, (2), the Pontryagin function takes the form:

$$K(\rho(\tau), \hat{O}(\tau), u(\tau)) = -i \text{Tr} \left\{ [\rho(\tau), \hat{O}(\tau)] (\hat{\sigma}_x + u(\tau) \hat{\sigma}_z) \right\}, \quad (6)$$

the evolution equation for $\hat{O}(\tau)$ coincides with (1):

$$\hat{O}(\tau'') = U_{\tau'', \tau'}(u) \hat{O}(\tau') U_{\tau'', \tau'}^\dagger(u), \quad (7)$$

and the boundary conditions read,

$$\hat{O}(T) = \hat{O}; \quad (8)$$

$$K(T) \begin{cases} = 0 & \text{if } T \text{ is unconstrained;} \\ \geq 0 & \text{in the case (4).} \end{cases} \quad (9)$$

Since the Pontryagin function (6) linearly depends on $u(\tau)$ the PMP can be satisfied in two ways,

- 1) The switching function $\frac{\partial}{\partial u(\tau)} K = -i \text{Tr} \left\{ [\rho(\tau), \hat{O}(\tau)] \hat{\sigma}_z \right\} \neq 0$. In this case $\tilde{u}(\tau) = u_{\max} \text{sign}(\frac{\partial}{\partial u(\tau)} K)$, and the corresponding section of the trajectory is called regular. It is clear that the optimal policy $\tilde{u}(\tau)$ is actively constrained, and relaxing the inequalities (3) will improve the optimization result. For this reason, the optimal trajectory containing the regular sections can not be kinematically optimal. An optimal process $\{\tilde{\rho}(\tau), \tilde{\hat{O}}(\tau), \tilde{u}(\tau)\}$ for which $\tilde{u}(\tau) = \pm u_{\max}$ everywhere except for the finite number of time moments is often referred as the bang-bang control.
- 2) It may happen that the switching function remains equal to zero over a finite interval of time. In this case, the corresponding segment of the trajectory is called singular, and the associated optimal control can be determined only from higher-order optimality criteria, such as the generalized Legendre-Clebsch conditions, Goh condition etc. [17, 29, 30]

Substituting (1) and (7) into (6) one can directly check that the Pontryagin function for problem (1) is constant along any extremal:

$$\forall \tau : K(\tau) = \tilde{K} \geq 0 \text{ on each extremal}, \quad (10)$$

where the strict inequality holds only if the constraint (4) is active, and

$$\forall \tau : K(\tau) \equiv 0 \text{ for any kinematically optimal solution.} \quad (11)$$

A. Singular extremals of the problem (1)

Every kinematically optimal solution $\tilde{u}(\tau)$ consist of a single singular subarc. Here we show that in the case of

the Landau-Zener system the converse is also true: every singular extremal $\tilde{u}(\tau)$ corresponding to inactive constraint (4) delivers the global kinematic extremum (maximum or minimum) to the problem (2). Indeed, let τ_1 be an arbitrary internal point of the singular trajectory. Then, the PMP states that:

$$\frac{\partial}{\partial u(\tau)} K(\tau) = -i \operatorname{Tr} \left\{ [\rho(\tau_1), \hat{O}(\tau_1)] U_{\tau, \tau_1}^\dagger(\tilde{u}) \hat{\sigma}_z U_{\tau, \tau_1}(\tilde{u}) \right\} \equiv 0 \quad (12)$$

for any τ such that $|\tau - \tau_1| < \epsilon$ and sufficiently small ϵ , and in particular:

$$-i \operatorname{Tr} \left\{ [\rho(\tau_1), \hat{O}(\tau_1)] \hat{\sigma}_z \right\} = 0. \quad (13a)$$

The two subsequent time derivatives of the equality (12) at $\tau = \tau_1$ give,

$$-i \operatorname{Tr} \left\{ [\rho(\tau_1), \hat{O}(\tau_1)] \hat{\sigma}_y \right\} = 0 \quad (13b)$$

$$-i \tilde{u}(\tau_1) \operatorname{Tr} \left\{ [\rho(\tau_1), \hat{O}(\tau_1)] \hat{\sigma}_x \right\} = 0. \quad (13c)$$

Equations (13) can be simultaneously satisfied only in the two cases:

$$[\rho(\tau), \hat{O}(\tau)] = 0; \quad (14a)$$

$$[\rho(\tau), \hat{O}(\tau)] = i\kappa \hat{\sigma}_x \text{ and } u(\tau) = 0 \text{ } (\kappa = \text{const} \neq 0). \quad (14b)$$

The condition (14a) is nothing but the criterion of the global kinematic extremum (maximum or minimum) for our two-level system. In other words, we just proved that all the extrema of the landscape $J(u)$ for the unconstrained Landau-Zener system except for the case of $u(t) \equiv 0$ are its global kinematic maxima and minima. This result was obtained in [18, 19].

The condition (14b) indicates that the only possible everywhere singular non-kinematic extremal of the problem (2) is $\tilde{u}(\tau) \equiv 0$ ($\tau \in [0, T]$). Eq. (6) implies that $K(\tau) = \kappa$ in this case. Thus, in view of (9), this extremal can appear only under the active pressure of the constraint (4).

B. Regular and mixed extremals of the problem (1)

According to the PMP and conditions (14), the generic non-singular extremal is the piecewise-constant function with n switchings of either bang ($u = \pm u_{\max}$) or bang-singular ($u = \pm u_{\max}, 0$) type where the singular arcs match (14b). For convenience, we will refer to extremals with (without) singular arcs as of type II (type I). We will use the subscript i (i.e. $\tilde{\tau}_i, \tilde{\rho}_i$ etc., $0 < i < n+1$) for the parameters related to the i -th control discontinuity (corner point). The durations of the right (left) adjacent arcs and the associated values of u will be labeled as $\tilde{\Delta}\tau_i$ ($\tilde{\Delta}\tau_{i-1}$) and \tilde{u}_i^+ (\tilde{u}_i^-). The subscripts $i=0$ and $i=n+1$ will be reserved for the parameters of the trajectory end-points. We will also sometimes use the notations ^sI and

^sII with index s denoting the number of times the control changes the sign.

We first address the properties of type I extremals. The necessary condition of the i -th corner point is given by eq. (13a). Combining it with (10) we get,

$$-i[\tilde{\rho}(\tilde{\tau}_i), \tilde{\hat{O}}(\tilde{\tau}_i)] = c_{i,1} \hat{\sigma}_x + c_{i,2} \hat{\sigma}_y, \quad c_{i,1}, c_{i,2} \in \mathbb{R}, \quad (15)$$

where $c_{i,1} = 0 (>0)$ when the constraint (4) is inactive (active) and the case $c_{i,1} \leq 0$ can result from the optimization with fixed T . Consider the adjacent $(i+1)$ -th bang arc. The PMP criterion (5) for its interior reads,

$$\tilde{u}(\tau)|_{\tau > \tilde{\tau}_i} = \arg \max_u \operatorname{Tr} [U_{\tau, \tilde{\tau}_i}(c_{i,1} \hat{\sigma}_x + c_{i,2} \hat{\sigma}_y) U_{\tau, \tilde{\tau}_i}^{-1} \hat{\sigma}_z] u, \quad (16)$$

which gives $\tilde{u}_i^+ = \frac{c_{i,2}}{|c_{i,2}|} u_{\max}$. If the $(i+1)$ -th arc ends with another corner point $\tilde{\tau}_{i+1}$ then it follows from (16) that,

$$\operatorname{Tr} [U_{\tilde{\tau}_{i+1}, \tilde{\tau}_i}(c_{i,1} \hat{\sigma}_x + c_{i,2} \hat{\sigma}_y) U_{\tilde{\tau}_{i+1}, \tilde{\tau}_i}^{-1} \hat{\sigma}_z] = 0. \quad (17)$$

Condition (17) can be reduced to the form,

$$c_{i,2} \sqrt{u_{\max}^2 + 1} = -c_{i,1} \tilde{u}_i^+ \tan(\tilde{\Delta}\tau_i \sqrt{u_{\max}^2 + 1}), \quad (18)$$

and resolved relative to $\tilde{\Delta}\tau_{i+1}$. Retaining the physically appropriate solutions consistent with eq. (16) we obtain,

$$\tilde{\Delta}\tau_{i+1} = \begin{cases} \tilde{\delta}\tau_i, & c_{i,1} < 0; \\ \pi \cos(\alpha) - \tilde{\delta}\tau_i, & c_{i,1} > 0, \end{cases} \quad (19)$$

where $\alpha = \arctan(u_{\max})$ and

$$\tilde{\delta}\tau_i = \arctan \left(\left| \frac{c_{2,i}}{c_{1,i} u_{\max}} \right| \sec(\alpha) \right) \cos(\alpha). \quad (20)$$

Note that $-i[\tilde{\rho}(\tilde{\tau}_{i+1}), \tilde{\hat{O}}(\tilde{\tau}_{i+1})] = c_{1,i} \hat{\sigma}_x - c_{2,i} \hat{\sigma}_y$, i.e.

$$c_{1,i+1} = c_{1,i}, \quad c_{2,i+1} = -c_{2,i}. \quad (21)$$

Since eqs. (19) and (20) do not depend on the sign of $c_{i,2}$ one obtains that durations of all interior bang segments are equal: $\forall i \geq 2, i \leq n : \tilde{\Delta}\tau_i = \tilde{\Delta}\tau$ (see Fig. 1a). Moreover, eq. (19) admits the estimate $\frac{\pi}{2} \cos \alpha \leq \tilde{\Delta}\tau \leq \pi \cos \alpha$ for the case of time-optimal problem with constraint (4).

Consider now the extremals of type II. Let $\tau \in (\tilde{\tau}_{j-1}, \tilde{\tau}_j)$ be the singular arc where the relations (14b) hold. If $\tilde{\tau}_j \neq \tilde{T}$ when it is the corner point between regular and singular arc. Suppose that there exists one more corner point $\tau_{j+1} > \tau_j$. Then it follows from eqs. (21) (19) and (17) that $\tilde{\Delta}\tau_j = \pi \cos \alpha$ and $U_{\tilde{\tau}_{j+1}, \tilde{\tau}_j} = -\hat{I}$, so that $\tilde{\rho}(\tilde{\tau}_{j+1}) = \tilde{\rho}(\tilde{\tau}_j)$. Using similar arguments, it is straightforward to derive the analogous result for possible corner points prior to τ_j . Thus, taking any 3-segment ‘‘anatz’’ extremal similar to that shown in Fig. 1b, one can construct an infinite family $\mathcal{F}^{[k]}(\tilde{u}(\tau))$ of $\Pi^{[k]}$ extremals ($k = k_1, k_2$) by randomly inserting k_1 and k_2 bang segments of the length $\pi \cos \alpha$ with $u = +u_{\max}$ and $u = -u_{\max}$

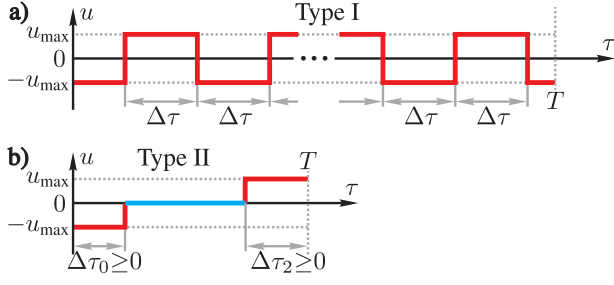


FIG. 1. Possible types of extremals $\tilde{u}(t)$ associated with nonkinematic optimal solutions and traps as well as the locally time-optimal kinematic optimal solutions.

into corner points of $\tilde{u}(\tau)$ or inside its singular arcs. It is clear that each family $\mathcal{F}^{[k]}(\tilde{u}(\tau))$ constitutes the connected set of solutions, and all the members have equal performances J . Thus, the properties of any type II extremal can be reduced to the analysis of the equivalent three-segment ^0II type or ^1II type extremal where all the positive and negative bang segments are merged into distinct continuous arcs separated by a singular arc.

The presented first-order analysis outlines the admissible profiles for optimal non-kinematic solutions (see Fig. 1). Moreover, by continuity argument (i.e. by considering the series of solutions with fixed $T \rightarrow T_{\text{opt}}$ from below), these profiles should embrace all possible types of the stationary points of the time-optimal problem (2),(4). It is worth stressing that the later include the globally optimal and everywhere singular kinematic solutions for which both segments with $u = \pm u_{\text{max}}$ and $u=0$ are singular. With this in mind, it is helpful to introduce the following terminological convention for the rest of the paper in order to preserve the integrity of the presentation while avoiding potential confusions: we will reserve the term “singular” exclusively for the segments of extremals at which $u=0$ whereas the segments with $u = \pm u_{\text{max}}$ will be always referred to as “bang” ones.

The reviewed results have several serious limitations. First, they do not allow to distinguish the globally time-optimal solution from the trap or saddle point. Second, they do not provide detailed *a priori* knowledge of the characteristic structural features of these stationary points (e.g. the expected type, number of switchings etc.) which is necessary to determine the topology of the landscape $J[u]$. These tasks require higher-order analysis, which is the subject of the next section.

III. DETAILED CHARACTERIZATION OF THE STATIONARY POINTS

In this section we will extensively use the geometrical arguments in our reasoning. To make the presentation more visual, it is useful to expand the states and observables in the basis of Pauli matrices and identity matrix \hat{I} : $\rho = \frac{1}{2}\hat{I} + \sum_{i=x,y,z} r_i \hat{\sigma}_i$, $\hat{O} = \frac{1}{2}\text{Tr}[\hat{O}]\hat{I} + \sum_{i=x,y,z} o_i \hat{\sigma}_i$. The

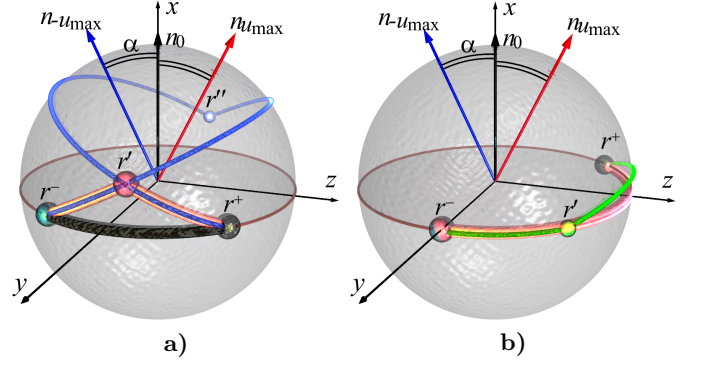


FIG. 2. (a) The case $r_y^- r_y^+ > 0$: Equatorial singular arc $r^- \rightarrow r^+$ (thick black line) is more time-effective than bang-bang extremal $r^- \rightarrow r' \rightarrow r^+$ (thick orange line). The extremal $r^- \rightarrow r'' \rightarrow r^+$ (thin blue curve) represents local extremum (trap). (b) The case $r_y^- r_y^+ < 0$: Equatorial singular arc $r^- \rightarrow r^+$ (thick orange line) is the suboptimal relative to the bang-singular extremal $r^- \rightarrow r' \rightarrow r^+$ (thin black line).

dynamics induced by eq. (1) corresponds to the rotation of the 3-dimensional Bloch vector $\vec{r} = \{r_x, r_y, r_z\}$ around the axis $\vec{n}_u \propto \{1, 0, u\}$ (note that the angle between $\vec{n}_{\pm u_{\text{max}}}$ and \vec{n}_0 is equal to α , see e.g. Fig. 2), and the optimization goal (2) is equivalent to the requirement to arrange the state vector \vec{r} in parallel to \vec{o} . In what follows we will often refer to the quantum states ρ as the endpoints r of vectors \vec{r} . Hereafter we will also assume that both r and o are renormalized (scaled) such that $|r| = |o| = 1$.

We start by taking a closer look at type II extremals and their singular arc(s) where $\tilde{u}(\tau) = 0$. According to criterion (14b), these arcs are always located in the equatorial plane $x=0$. The following proposition indicates that such arcs may represent the time-optimal solution at any values of u_{max} (see Appendix A for proof):

Proposition 1. *The shortest type II singular trajectory connecting any two “equatorial” points $\vec{r}^- = \{0, r_y^-, r_z^-\}$ and $\vec{r}^+ = \{0, r_y^+, r_z^+\}$ (see Fig. 2) will represent the (globally) time-optimal solution if $r_y^- r_y^+ > 0$, $(r_z^+ - r_z^-) r_y^- > 0$ and the saddle point otherwise.*

Since all ^sII extremals can be reduced to the effective 3-section ansatz (see the end of the previous section) Proposition 1 has the evident corollary:

Proposition 2. *All singular arcs of the locally optimal type II extremals are located in the same semi-space $y > 0$ or $y < 0$, and their total duration can not exceed $\pi/2$.*

For further analysis we need the following generic necessary condition of the time optimality:

Proposition 3. *If the type I extremal $\{\tilde{u}(\tau), \tilde{r}(\tau)\}$ is locally time-optimal then each of its corner points \tilde{r}_i satisfies the inequality:*

$$\tilde{u}_i^- \tilde{r}_{i,x} \tilde{r}_{i,y} \geq 0. \quad (22)$$

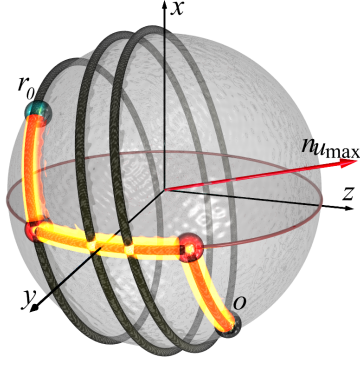


FIG. 3. The globally time-optimal ^0II type trajectory $\tilde{u}_{\text{anz}}(\tau)$ (thick bright yellow curve) and the locally time-optimal trapping solution (black curve) of the $\mathcal{F}^{[3]}(\tilde{u}_{\text{anz}}(\tau))$ family connecting the points $r_0 \propto \{1, 1, -1\}$ and $o \propto \{-1, 1, 1\}$.

Qualitatively, Proposition 3 states that the projections of optimal trajectories on the xz -plane are always "V"-shaped at the corner points \tilde{r}_i with $\tilde{r}_{i,x} > 0$ and "Λ"-shaped otherwise (here we assume that the x -axis is oriented vertically, like in Fig. 2).

Proposition (3) allows to substantially narrow down the range of the type II candidate trajectories:

Proposition 4. *Any type ^sII extremal with $s > 0$ containing the interior bang arc of duration $\Delta\tau > \pi \sec \alpha$ is the saddle point for the time-optimal control.*

In other words, all the type $^s\text{II}|_{s>0}$ locally time-optimal solutions reduce to the 3-piece ansatz shown in Fig. 1b where two regular arcs of duration $\tilde{\Delta}\tau_0, \tilde{\Delta}\tau_2 < \pi \sec \alpha$ "wrap" the singular section where $u=0$. Accordingly, the number of control switchings is bounded by $n_{\text{II}} \leq 2$.

The properties of the ^0II type extremals are richer:

Proposition 5. *Suppose that the ^0II type extremal $\tilde{u}(\tau)$ is the member of family $\mathcal{F}^{[k]}(\tilde{u}_{\text{anz}}(\tau))$, and its ansatz $\tilde{u}_{\text{anz}}(\tau)$ has nonzero durations $\tilde{\Delta}\tau_0$ and $\tilde{\Delta}\tau_2$ of opening and closing bang segments. Then $\tilde{u}(\tau)$ is locally optimal iff $\tilde{u}_{\text{anz}}(\tau)$ is locally optimal*

(for proof see Appendix D).

The analysis of type I extremals is somewhat more complicated. We begin by determining the loci of corner points \tilde{r}_i on the Bloch sphere. Denote as $\theta = 2\tilde{\Delta}\tau \sec \alpha$ the rotation angles on the Bloch sphere associated with the inner bang sections of the type I extremals. Note that it follows from (19), (20) that $\pi < \theta < 2\pi$ in the case of time-optimal control problem.

Proposition 6. *All the corner points \tilde{r}_i of any locally optimal type I solution $\tilde{u}(\tau)$ of problem (2), (3) are located on the circular intersections of the Bloch sphere with the two planes $\lambda_{\pm 1}$ (see Fig. 4):*

$$\tilde{r}_i = \left\{ \text{sign}(\tilde{u}_i^+) \sin(\gamma_i) \sin\left(\frac{\xi}{2}\right), -\sin(\gamma_i) \cos\left(\frac{\xi}{2}\right), \cos(\gamma_i) \right\} \quad (23)$$

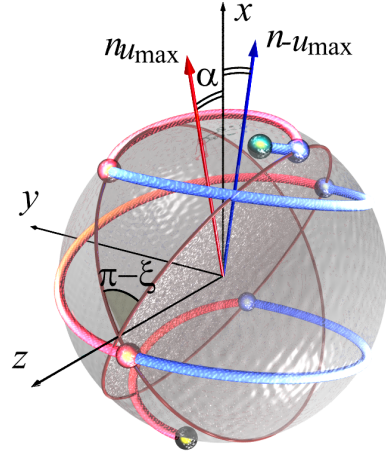


FIG. 4. Illustration of the statement of Proposition 6. The thick colored curve depicts the band-bang extremal. Its red and blue segments correspond to $u = \max$ and $u = -\max$. All interior corner points (red and blue balls) lie on two circles (associated with switchings $u_{\max} \rightarrow -u_{\max}$ and $-u_{\max} \rightarrow u_{\max}$, correspondingly) whose planes $\lambda_{\pm 1}$ intersect along the z -axis.

Here $\xi = -2 \arctan\left(\frac{u_{\max}}{2} \tan\left(\frac{\theta}{2}\right) \cos(\alpha)\right)$ is the dihedral angle between the planes $\lambda_{\pm 1}$, and $\gamma_{i+1} = \gamma_i + i\eta$, where $\eta = -2 \arctan\left(\frac{\sin(\frac{\theta}{2})}{\sqrt{u_{\max}^2 + \cos^2(\frac{\theta}{2})}}\right)$.

Proposition 7. *Denote $q_i = q(\gamma_i) = \cot^2(\gamma_i) - \cot^2(\frac{\eta}{2})$ ($i=1, \dots, n$). The set $\{q_i\}$ associated with any locally time-optimal extremal $\tilde{u}(t)$ contains at most one negative entry q' , and $|q'| = \min(|\{q_i\}|)$.*

The proofs of the above two propositions are given in Appendix E.

To use Proposition 7 it is convenient to introduce the parameters ζ_i through, $\zeta_1 = \gamma_1 + \frac{\pi}{2}(1 - \text{sign}(u_1^+))$, $\zeta_{i+1} = \zeta_i + i(\pi + \eta)$. It is evident that $q(\gamma_i) = q(\zeta_i)$. The relation between the sign of q_i and the index i of the corner point can be illustrated by associating each q_i with the point on the unit cycle whose position is specified by ζ_i , as shown in Fig. 5. One can see that the maximal number n_{\max} of sequential parameters q_i having at most one negative term can not exceed $\frac{\pi + |\eta|}{\pi - |\eta|} + 1 \leq \frac{\pi}{\alpha}$, i.e.,

Proposition 8. *Type I locally optimal extremals can have at most $\frac{\pi}{\alpha}$ switchings.*

This helpful upper bound was first obtained by Agrachev and Gamkrelidze [26]. As shown in Appendix F, we can further refine this result via more detailed inspection of the criterion $|q'| = \min(|\{q_i\}|)$:

Proposition 9. *$n_{\text{I}, \max} \leq 2$ if $u_{\max} > \sqrt{1 + \sqrt{2}}$*

(The latter roughly corresponds to $\alpha > 1$).

The analysis in this section so far is equally valid for both global and local extrema (traps) of optimal control. It is clear that any globally time-optimal type II solution

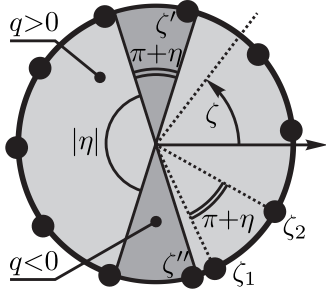


FIG. 5. Signs of parameters $q(\zeta)$ as function of ζ . Black dots indicate the values $\zeta = \zeta_i$ associated with i -th corner point.

includes at most 2 corner points that separate the central singular section from the outside regular arcs (see Fig. 1b). The case of type I solutions is not as evident. The following propositions impose more stringent necessary criteria on the globally time-optimal extremals (see Appendices G and H for proofs).

Proposition 10. Any corner point \tilde{r}_i such that $q(\gamma_i) < 0$ must be either the first or the last corner point of the globally time optimal solution, so that the total number of switchings $n_{I,\max} \leq \frac{\pi}{2\alpha} + 1$.

Proposition 11. The corner points \tilde{r}_i of any globally optimal solution of type I satisfy the inequality:

$$\min(0, \tilde{r}_{0,x}, \tilde{r}_{n+1,x}) < \tilde{r}_{i,x} < \max(0, \tilde{r}_{0,x}, \tilde{r}_{n+1,x}), \quad (24)$$

where $\tilde{r}_{0,x}$ and $\tilde{r}_{n+1,x}$ are the trajectory endpoints.

Proposition 11 can be used to establish the following more accurate upper bound on the number of switchings (see Appendix I for proof).

Proposition 12. The number of corner points of the globally time-optimal type I solution $\tilde{u}(\tau)$ is bounded by the following inequalities:

$$left = n_I \leq \left\lceil \max\left(\frac{\arccos(\frac{\tilde{r}_x^-}{\tilde{r}_x^+})}{|2 \arctan(\frac{u_{\max}}{\tilde{r}_x^+})|}, \frac{\pi}{|2 \arctan(\frac{u_{\max}}{\tilde{r}_x^+})|}\right) + 1 \right\rceil \quad \text{if } \tilde{r}_x^- \tilde{r}_x^+ < 0; \quad (25a)$$

$$\min\left(\frac{\arccos(\frac{\tilde{r}_x^-}{\tilde{r}_x^+})}{|2 \arctan(\frac{u_{\max}}{\tilde{r}_x^+})|} + 3, \frac{\pi}{|4 \arctan(\frac{u_{\max}}{\tilde{r}_x^+})|}\right) + 1 \quad \text{if } \tilde{r}_x^- \tilde{r}_x^+ > 0, \quad (25b)$$

where \tilde{r}^+ and \tilde{r}^- are new notations for the trajectory endpoints \tilde{r}_0 and \tilde{r}_{n+1} , such that $|\tilde{r}_x^+| \geq |\tilde{r}_x^-|$.

Let us denote $\phi_\xi = |\theta_{r_0,\xi} - \theta_{o,\xi}|$ ($\xi = x, z$), where $\theta_{r,\xi}$ is the angle between the axes \vec{e}_ξ and \vec{r} . One can geometrically show that the maximal possible change $\Delta\theta_{r,\xi}^{\max}$ in $\theta_{r,\xi}$ generated by rotation around any of the axes $\vec{n}_{\pm u_{\max}}$ is $\Delta\theta_{r,x}^{\max} = 2\alpha$ and $\Delta\theta_{r,z}^{\max} = \pi - 2\alpha$ (see Fig. 6). This fact allows us to establish the following lower bounds on the number of corner points:

Proposition 13. The minimal number of corner points in locally time-optimal solutions reaching the global maximum of J is bounded by the inequalities:

$$n \geq \frac{|\arcsin(r_{0,x}) - \arcsin(o_x)|}{2 \arctan(u_{\max})} - 1; \quad (26a)$$

$$n_1 \geq \frac{|\arcsin(r_{0,z}) - \arcsin(o_z)|}{2 \operatorname{arccot}(u_{\max})} - 1; \quad (26b)$$

It is worth stressing that the bound (26b) is valid only for type I solutions.

Combination of the upper bounds on n imposed by Propositions 4 and 10 with inequalities (26) leads to the following conclusion:

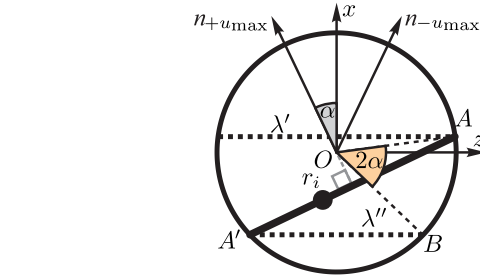


FIG. 6. Geometrical calculation of the value of $\Delta\theta_{r,x}^{\max}$. Rotation $S_{\vec{n}_{-u_{\max}}}$ around vector $\vec{n}_{-u_{\max}}$ transfers any point r_i on Bloch sphere into new point in AA' plane. The x -coordinate of this new point is bounded by planes λ' and λ'' . Thus, the associated change in $\theta_{r,x}$ is less than $\angle AOB = 2\alpha$.

Proposition 14. The globally time-optimal solution(s) of problem (2) is of type I if

$$\phi_x = |\arcsin(r_{0,x}) - \arcsin(o_x)| > 4\alpha \quad (27a)$$

and of type II if

$$\phi_z = |\arcsin(r_{0,z}) - \arcsin(o_z)| > \left\lceil \frac{\pi}{2\alpha} + 2 \right\rceil (\pi - 2\alpha). \quad (27b)$$

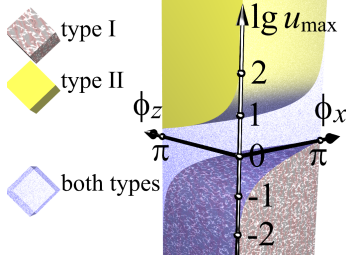


FIG. 7. Distribution of types of globally optimal solutions according to Proposition 14. Note that the admissible values of ϕ_x and ϕ_z are restricted by inequality $\phi_x + \phi_z \leq \pi$.

Note that this estimate can be further detailed if combined with the refined upper bounds stated in Proposition 12. The statement of Proposition 14 is graphically illustrated in Fig. 7 which clearly shows that the type I and type II solutions are dominant in the opposite limits of tight and loose control restriction $u_{\max} \rightarrow 0$ and $u_{\max} \rightarrow \infty$, correspondingly. Neither type, however, totally suppresses the other one at any finite positive value of u_{\max} . This coexistence sets the origin for the generic structure of suboptimal solutions (traps), whose analysis will be the subject of next two sections.

IV. TRAPS IN TIME-OPTIMAL CONTROL

The globally time-optimal solution (hereafter denoted as \tilde{u}^{opt}) of the problem (2) can be supplemented by a number of trapping suboptimal solutions \tilde{u} (characterized by $\tilde{J} < \tilde{J}^{\text{opt}}$ and/or $\tilde{T} > \tilde{T}^{\text{opt}}$) which are however optimal with respect to any infinitesimal variation of $\tilde{u}(\tau)$ and T . In particular, Proposition (2) implies that each locally optimal solution of type 0II gives rise to the infinite family of traps of the form shown in Fig. 3. In what follows, we will call such traps as “perfect loops”. Proposition 1 indicates that the perfect loops may exist at any value of u_{\max} . Nevertheless, their presence does not stipulate sufficient additional complications in finding the globally optimal solution by gradient search methods. Indeed, these “simple” traps can be identified at no cost by the presence of the continuous bang arc of the duration $\Delta\tau_i \geq \pi \sec(\alpha)$. Moreover, one can easily escape any such trap by inverting the sign of the control $u(\tau)$ at any continuous subsegment of this arc of duration $\pi \sec(\alpha)$ or by removing the respective time interval from the control policy.

For this reason, the primary objective of this section is to investigate the other, “less simple” types of traps which can be represented by type I and $^s\text{II}|_{s>0}$ suboptimal extremals. Propositions 8, 10, 12, and 13 show that the number of switchings n in such extremals is always bounded (at least by π/α). Thus, the maximal number of such traps is also finite and decreases with increasing u_{\max} . It will be convenient to loosely classify the traps into the “deadlock”, “loop” and “topological” ones as

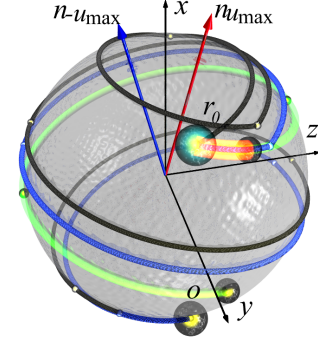


FIG. 8. Globally optimal solution (blue line), deadlock traps (light-red and green lines) and loop trap (black line) for the time-optimal control problem (2),(3),(4) with $u_{\max} = \frac{1}{4}$, $r(0) = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\}$, (big emerald dot) and $o = \{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\}$ (big black-yellow dot). Small dots indicate the positions of corner points. The parameters of extremals are listed in the table:

extremal	$\text{sign}(\tilde{u}_1^-)$	n	$\tilde{\Delta\tau}_1$	$\tilde{\Delta\tau}$	$\tilde{\Delta\tau}_{n+1}$
red	+	0	0.23	-	-
green	-	2	0.88	1.52	0.88
blue	+	4	0.33	1.78	0.33
black	-	5	1.15	1.72	0.57

follows. The first two kinds of traps are represented by type I extremals. The deadlock traps are defined by inequalities $\tilde{J} < \tilde{J}^{\text{opt}}$ $\tilde{T} < \tilde{T}^{\text{opt}}$. They usually also satisfy the inequalities $n < n^{\text{opt}}$. Their existence is mainly related to the fact that the distance to the destination point o for most of extremals non-monotonically changes with time. The trajectory of the loop trap has the intersection with itself other than the perfect loop. These solutions require longer times $\tilde{T} > \tilde{T}^{\text{opt}}$ and typically also larger numbers of switchings $n > n^{\text{opt}}$ in order to reach the kinematic extremum $\tilde{J} = \tilde{J}^{\text{opt}}$. Finally, the topological traps are associated with extremals of the type distinct from the type of the globally optimal solution. Of course, real traps can combine the features of all these three kinds.

Examples of the deadlock and loop traps are shown in Fig 8. In this case the globally time optimal solution with $n^{\text{opt}} = 4$ is accompanied by two deadlock traps and two degenerate loop traps corresponding to $n = 5$ (only one is shown; the remaining solution can be obtained via subsequent reflections of the black trajectory relative to the yz and xy -planes). At the same time, no traps exist for $n = 1, 3$ and $n > 5$.

The bang-bang extremal $r^- \rightarrow r'' \rightarrow r^+$ (blue curve) in Fig. 2a provides another example of the loop trap that is also the topological trap relative to type II optimal trajectory $r^- \rightarrow r^+$ (the specific parameters used in this example are: $u_{\max} = \frac{1}{2}$, $r^- = r_0 \propto \{0, 1, -\frac{1}{2}\}$, $r^+ = o \propto \{0, 1, 1\}$). In general, once the endpoints r^- and r^+ satisfy the conditions of Proposition 1, the time-optimal solution remains the same type II trajectory even in the limit $u_{\max} \rightarrow 0$, where the most time optimal trajectories are of type I (see Proposition 14 and Fig. 7). Moreover the traps of the shown form will exist for any value of

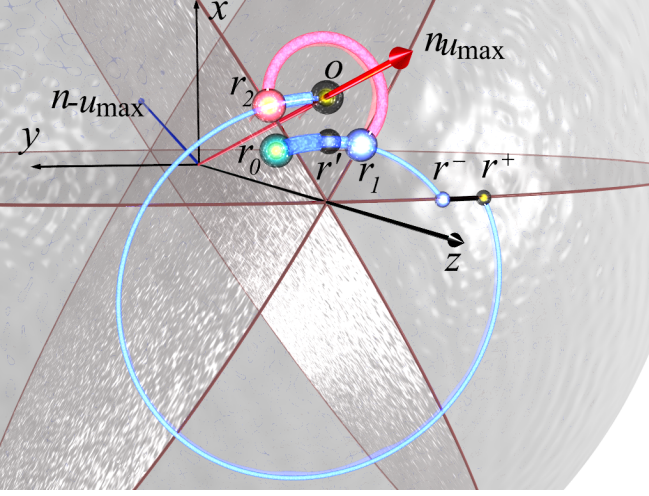


FIG. 9. The optimal solution (medium-thick trajectory $r_0 \rightarrow r_1 \rightarrow r_2 \rightarrow o$), topological trap (thin trajectory $r_0 \rightarrow r^- \rightarrow r^+ \rightarrow o$) and deadlock trap (thick trajectory $r_0 \rightarrow r'$) for the time-optimal control problem (2),(3),(4) with $u_{\max}=8$, $r(0) \propto \{\frac{1}{2}, \frac{1}{2}, u_{\max}\}$, $o \propto \{1, 0, u_{\max}\}$. The segments colored blue/black/red correspond to $u(\tau) = -u_{\max} \setminus 0 \setminus +u_{\max}$ and are associated with rotations about the axes $\vec{n}_{-u_{\max}} \setminus \vec{e}_x \setminus \vec{n}_{+u_{\max}}$. The durations $\Delta\tau_i$ of the consequent bang arcs are summarized in the table:

extremal	type	n	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$
deadlock trap	I	0	0.020	-	-
optimal solution	I	2	0.0327	0.262	0.017
topological trap	II	2	0.075	0.031	0.324

$$u_{\max} < \sqrt{4 - (r_z^- + r_z^+)^2} / |r_z^- - r_z^+|.$$

Another generic example of the traps of all three types can be straightforwardly constructed in the case $u_{\max} \gg 1$ (see Fig. 9) by selecting $o \propto \{1, 0, u_{\max}\}$ and choosing the initial state in vicinity of $z=1$: $r_0 \propto \{c_1, c_2, u_{\max}\}$, where $0 < c_1 < 1$ and c_2 is any sufficiently small number. Although the vast majority of time-optimal solutions are of type II in the limit $u_{\max} \rightarrow \infty$ (see Proposition 14), for this special choice the optimal solution is of type I for any finite value of u_{\max} whereas the complementary type II extremal represents the topological trap. In the case $c_2 < 0$, there also exist a deadlock trap structurally similar to the ones shown in Fig. 8.

These observations lead to the following key proposition:

Proposition 15. *For any value of u_{\max} there exist initial states ρ_0 , observables \hat{O} and locally time-optimal control policy $\tilde{u}(\tau)$ whose constitutes the non-simple traps of time optimal control problem (2),(3),(4).*

V. TRAPS IN THE FIXED-TIME OPTIMAL CONTROL

Consider the problem (2), (3) where the control time T is fixed. Specifically, we will be interested in the case

$$T = \text{const} \gg \frac{\pi^2}{\alpha} \quad (28)$$

when the kinematically optimal solutions exist for any given ρ_0 and \hat{O} . We again will exclude the class of perfect loop traps from the analysis for the same reasons as in the previous section. Intuitively one can expect that the probability of trapping in the local extrema (other than perfect loops) should be small at large T . However, it is not clear if there exists such value of T that the functional (2) will become completely free of such traps.

To answer this question, note that in line with the analysis given in Sec. II any trap should be represented by either type I or type II extremal. However, the maximal number of switchings is no longer limited by inequalities similar to Proposition 8. At the same time, Proposition 6 remains applicable (see *Remark 1* in Appendix E). Recall that its proof is based on introduction of the “sliding” variations $\delta\gamma_i$ which shift the angular positions of the “images” of corner points on the diagram of Fig. 5 (see Appendix E). The explicit expression for the “sliding” variation around the i -th corner point up to the third order in the associated control time change $\delta\tau_i$ is given by eq. (E4). By definition, if the trajectory $\tilde{u}(\tau)$ is type I trap, then no admissible control variation δu can improve the performance index (2). Consider the subset Ω of such variations composed of infinitesimal sliding variations $\delta\gamma_i$ that preserve the total control time T . Then, the necessary condition of trap $\tilde{u}(\tau)$ is absence of the non-uniform sliding variation $\delta u(\tau) \in \Omega$ that leaves the trajectory endpoint r_{n+1} intact. Indeed, the trajectory associated with varied control $\tilde{u} + \delta u$ would deliver the same value of the performance index but at the same time is not the locally optimal solution (since it is no longer the type I extremal) which implies that \tilde{u} is not locally optimal.

Using (E4) the stated necessary condition can be rewritten as the requirement of definite signature of the quadratic form (E6), where the parameters q_i were introduced in Proposition (7). The necessary condition of the sign definiteness is that all (probably except one) parameters q_i are either non-positive or non-negative. Using Fig. 5 one can see that in the case of long T only the second option can be realized with $\eta \simeq 0$, $\eta \simeq -\frac{\pi}{2}$ and $\eta \simeq -\frac{\pi}{3}$ (the case $\eta \simeq -\pi$ must be eliminated because it implies $u_{\max}=0$). One can show that the last two variants lead to saddle points rather than to the local extrema. The remaining case $\eta \simeq 0$ leaves the two options $\theta \simeq 0$ and $\theta \simeq 2\pi$. The last option corresponds to positive constant $c_{i,1}$ in (15), which indicates the possibility of increasing J via monotonic “stretching” the time: $T \rightarrow T + \delta T(T)$, $u(\tau) \rightarrow u(\tau - \delta T(\tau))$, where $\delta T(\tau)$ is an infinitesimal positive monotonically increasing function. At the same time, the associated parameters q_i are all negative, so there ex-

ists the combination of variations $\delta\tau$ of arcs durations $\Delta\tau$ which will result in achieving the same value of the performance index at shorter time. Thus, we can conclude that it is also possible to increase J at fixed time T via proper combination of these two variations, so the variant $\theta \simeq 2\pi$ should be dismissed as a saddle point. Only the remaining choice $\theta \simeq 0$ is consistent with an arbitrary number of q_i of the same sign. However, in this case the length of each bang arc also reduces to zero. As result, the maximal duration of such optimal trajectories is limited by the inequality $T \lesssim \pi$.

This analysis leads us to remarkable conclusion:

Proposition 16. *The fixed-time optimal control problem (2) is free of non-simple traps for sufficiently long control times T .*

The spirit of this conclusion is in line with the results of numerical simulations performed in [19]. With this, it is worth recalling that the general time-fixed problem may have a variety of perfect loop traps for any value of u_{\max} and, thus, is not trap-free in the strict sense. These traps were missed in the simulations in [19] due to the specifics of numerical optimization procedure.

VI. SUMMARY AND CONCLUSION

All stationary points of the time optimal control problem and all saddles and local extrema of the fixed-time optimal control problem are represented by the piecewise-constant controls of types I and II sketched in Fig. 1 (the associated characteristic trajectories $\rho(\tau)$ on the Bloch sphere are shown in Figs. 4 and 3, correspondingly). We systematically explored the anatomy of stationary points of each type. Specifically, we identified the locations and relative arrangements of corner points on the Bloch sphere (propositions 2, 3, 6, 7, 10, 11) and estimated their total number (propositions 8, 9, 10, 12, 13). These characteristics together with propositions 1, 4, 5 and 14 allow to determine whether the given extremal is a saddle point or locally optimal solution, and also to predict the shape of globally optimal solution. The presented results (except Proposition 8) substantially generalize and refine the estimates obtained in previous studies [24, 25]. Moreover, this study, to our knowledge, is the first example of a systematic analytic exploration of the overall topology of the quantum landscape $J[u]$ in the presence of constraints on the control u and for the arbitrary initial quantum state ρ_0 and observable \hat{O} . In particular, we distinguished 4 categories of traps tentatively called deadlock, topological, loop and perfect loop traps. The landscape can contain an infinite number of perfect loops whereas the number of traps of other types is always finite. Among them the number of deadlock traps and loops decreases with increasing value of the constraint u_{\max} in eq. (4). Nevertheless, we have shown by an explicit example that the traps of all categories can simultaneously complicate the landscape $J[u]$ of the

time-optimal control problem regardless of the value of u_{\max} . So, this is the case where the intuitive attempt to “extrapolate” the conclusions based on analysis of the case of unconstrained controls totally fails.

The fixed-time control problem is more intriguing. On one hand we formally showed that it is impossible to completely “flatten” all the traps in this case by increasing the value of u_{\max} . This result is in line with generic experience concerning the optimal control in technical applications. However, if the control time is long enough (specifically, if $T \gg \pi^2 / \arctan u_{\max}$) the only traps which can survive are perfect loops. These traps can be easily escaped via simple modification of any gradient search algorithm at virtually no computational cost. Thus, the quantum landscape appears as trap-free from practical perspective, which supports the common viewpoint in quantum optimal control community. Since the controlled two-level system is the benchmark for quantum information processing this finding is relevant for efficient optimal control synthesis in a variety of experiments on cold atoms, Bose-Einstein condensates, superconducting qubits etc.

The key methodological feature of the presented derivations is introduction of the sliding variations which makes it possible to extensively rely on highly visual and intuitive geometrical arguments. For this reason, we believe that the mathematical aspect of the paper constitutes instructive introduction into high-order analysis of optimal processes.

Appendix A: Proof of Proposition 1

Here we consider the case $r_y^-, r_y^+ > 0$. The case $r_y^-, r_y^+ < 0$ can be treated similarly. Simple geometrical analysis leads to the following expression for the travel time difference δT between bang-bang (orange) and “equatorial” (black) trajectories shown in Fig. 2a:

$$\begin{aligned} \delta T_a = & \cos(\alpha) \left(\arcsin \left(\frac{\frac{\delta_z}{2} \sec(\alpha) - \cos(\alpha) r_z^+}{\sqrt{1 - \sin^2(\alpha) r_z^{+2}}} \right) + \right. \\ & \arcsin \left(\frac{\cos(\alpha) r_z^+}{\sqrt{1 - \sin^2(\alpha) r_z^{+2}}} \right) - \arcsin \left(\frac{\cos(\alpha) r_z^-}{\sqrt{1 - \sin^2(\alpha) r_z^{-2}}} \right) + \\ & \left. \arcsin \left(\frac{\frac{\delta_z}{2} \sec(\alpha) + \cos(\alpha) r_z^-}{\sqrt{1 - \sin^2(\alpha) r_z^{-2}}} \right) \right) - \arcsin(r_z^+) + \arcsin(r_z^-), \end{aligned} \quad (A1)$$

where $\delta_z = r_z^+ - r_z^-$. Let us fix one of the endpoints r^\pm and vary the position of another one. Note that $\delta T_a|_{\delta_z=0} = 0$

for any admissible value of r_z^\pm . Furthermore,

$$\pm \frac{d\delta T_a}{dr_z^\pm} = \frac{(1-r_z^{\pm 2}) \left(\sqrt{1 - \frac{r_z^\pm \delta_z}{1-r_z^{\pm 2}}} - \sqrt{1 - \frac{r_z^\pm \delta_z + \frac{\delta_z^2}{4} \sec^2 \alpha}{1-r_z^{\pm 2}}} \right)}{\left(\csc^2 \alpha - r_z^{\pm 2} \right) \sqrt{1 - r_z^\pm \delta_z - r_z^{\pm 2} - \frac{\delta_z^2}{4} \sec^2 \alpha}} > 0. \quad (\text{A2})$$

This allows to conclude that $\delta T_a > 0$ for any $\delta_z > 0$ which finishes the proof of Proposition for the case $r_y^- r_y^+ > 0$.

Consider now the case $r_y^- r_y^+ < 0$. For clarity, we will assume that $r_y^- > 0$, $r_z^- < r_z^+$ (see Fig. 2b). The remaining cases can be analyzed similarly. The time difference δT_b between “equatorial” (black) and the green trajectories and its derivative with respect to the position of the endpoint r_z^+ read:

$$\delta T_b = \arccos(r_z^+) - \cos(\alpha) \arccos \left(\frac{r_z^+ \cos(\alpha)}{\sqrt{1 - r_z^{+2} \sin^2(\alpha)}} \right); \quad (\text{A3})$$

$$\frac{\partial}{\partial r_z^+} \delta T_b = - \frac{2\sqrt{1 - r_z^{+2} \sin^2(\alpha)}}{r_z^{+2} \cos(2\alpha) - r_z^{+2} + 2}. \quad (\text{A4})$$

These expressions show that $\delta T_b|_{r_z^+ = 1} = 0$ and that $\frac{\partial}{\partial r_z^+} \delta T_b > 0$ for any admissible value of r_z^+ . Thus, $\delta T_b > 0$ which proves Proposition for the case $r_y^- r_y^+ < 0$.

Appendix B: Proof of Proposition 3

The proof is based on explicit construction of the second-order McShane’s (needle) variation of the control $\tilde{u}(\tau)$ which decreases \tilde{T} if the inequality (22) is violated. Choose arbitrary infinitesimal parameter $\delta\tau^- \rightarrow 0$ and denote $r_i^- = \tilde{r}(\tilde{\tau}_i - \delta\tau^-)$. Under assumptions of Proposition it is always possible (except for the trivial case $\tilde{r}_{i,y} = 0$) to choose another small parameter $\delta\tau^+$ such that the state vector $r_i^+ = \tilde{r}(\tilde{\tau}_i + \delta\tau^+)$ obeys the equality: $r_{i,x}^- = r_{i,x}^+$. It is evident that the Bloch vector $r_{i,x}$ can also reach $r_{i,x}^+$ in the course of free evolution with $u=0$ after certain time $\delta\tau^0$. If we require that $\delta\tau_i^+, \delta\tau_i^0|_{\delta\tau_i^- \rightarrow 0} = 0$ then both τ_i^+ and τ_i^0 are uniquely defined by $\delta\tau_i^-$:

$$\begin{aligned} \delta\tau_i^+ &= \frac{\delta\tau_i^- (\tilde{r}_{i,y} + 2\delta\tau_i^- \tilde{r}_{i,z})}{\tilde{r}_{i,y}} + o(\delta\tau_i^{-2}); \\ \delta\tau_i^0 &= \frac{2\delta\tau_i^- (\delta\tau_i^- (\tilde{u}_i^- \tilde{r}_{i,x} + \tilde{r}_{i,z}) + \tilde{r}_{i,y})}{\tilde{r}_{i,y}}, \end{aligned} \quad (\text{B1})$$

and thus, $\delta\tau_i^0 - \delta\tau_i^+ - \delta\tau_i^- = 2\tilde{u}_i^- (\delta\tau_i^-)^2 \tilde{r}_{i,x} / \tilde{r}_{i,y}$. The latter quantity should be nonnegative for the locally time-optimal solution which leads to eq. (22).

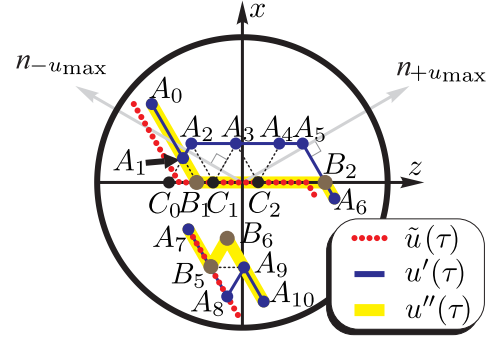


FIG. 10. Projections of the characteristic pieces of the original, varied and reduced trajectories $\tilde{r}(\tau)$, $r'(\tau)$ and $r''(\tau)$ on the xz plane (it is assumed that y -components of all shown parts of trajectories are greater than zero). The color associations are indicated in the inset.

Appendix C: Proof of Proposition 4

Consider any type $s\Pi$ extremal with $s > 0$. By definition, such extremals must contain at least one interior bang segment $\tau \in [\tilde{\tau}_i, \tilde{\tau}_{i+1}]$ of length $\tilde{\Delta}\tau_i = m\pi \cos(\alpha)$ ($m \in \mathbb{N}, 0 < \tilde{\tau}_i, \tilde{\tau}_{i+1} < T$). Since $\tilde{\tau}_i = \tilde{\tau}_{i+1}$ both the value of the performance index J and duration T will not change if this segment will be “translated” in arbitrary new point $\tilde{\tau}(\tau'_i(\kappa))$ of extremal via the following continuous variation $\tilde{u}(\tau) \rightarrow u(\kappa, \tau)$ ($-\tau_i < \kappa < T - m\pi \cos \alpha$):

$$u(\kappa, \tau) = \begin{cases} \tilde{u}(\tau), & \tau < \tilde{\tau}_i + \frac{\kappa - |\kappa|}{2} \vee \tau > \tilde{\tau}_{i+1} + \frac{\kappa + |\kappa|}{2}; \\ \tilde{u}_i^+, & \tilde{\tau}_i + \kappa < \tau < \tilde{\tau}_{i+1} + \kappa; \\ u(\tau - \tilde{\Delta}\tau_i) & \text{otherwise,} \end{cases} \quad (\text{C1})$$

where $\tau'(\kappa) = \tilde{\tau}_i + \kappa + \frac{1}{2}(1 + \frac{\kappa}{|\kappa|})\tilde{\Delta}\tau_i$.

Suppose that $\tilde{u}(\tau)$ is locally time-optimal solution. Then all the family of control policies $\{u(\kappa, \tau), r(\kappa, \tau)\}$ should be locally time-optimal too. Since $s > 0$ it is always possible to select the value $\kappa = \kappa_0$ such that $\tilde{r}(\tau'(\kappa_0))$ is interior point of the bang arc with $\tilde{u}(\tau'(\kappa_0)) = -\tilde{u}_i^+$ and $\tilde{r}_x(\tau'(\kappa_0)) \neq 0$. However, the resulting trajectory $r(\kappa_0, \tau)$ is both Λ - and V -shaped in the neighborhood of point $r(\tilde{\tau}_i + \kappa_0) = r(\tilde{\tau}_{i+1} + \kappa_0)$. According to Proposition (3) such trajectory can not be time-optimal. The obtained contradiction finishes the proof.

Appendix D: Proof of Proposition 5

Let $u'(\tau)$ be the control strategy obtained via arbitrary McShane variation $\delta u(\tau)$ of the control $\tilde{u}(\tau)$. Let us show that $u'(\tau)$ is less time efficient than some member $u''(\tau)$ of the control family $\mathcal{F}^{[k]}(u''^{\text{anz}}(\tau))$ with the same k but perhaps the different ansatz u''^{anz} . For this we will need the following lemma which is complementary to Propositions 1 and 3:

Lemma 1. Suppose that $r'(\tau')$ is junction point of two bang arcs of the trajectory $u(\tau)$ such that $r'_x=0$. Consider any two points $r^-(\tau^-)$ and $r^+(\tau^+)$ ($\tau^- < \tau' < \tau^+$) on adjacent arcs such that $r_x^- = r_x^+$ and the complete segment $r'_y r_y(\tau) > 0$ for any $\tau \in (\tau^-, \tau^+)$. Denote as $\widehat{\Delta\tau}$ the minimal duration of free evolution ($u=0$) required to reach r^+ starting from r^- . Then, $\widehat{\Delta\tau} > \tau^+ - \tau^-$.

Since $\tilde{u}_{\text{anz}}(\tau)$ is locally optimal by assumption it is sufficient to consider the variations of the $\delta u(\tau)$ which do not involve the vicinities of the trajectory endpoints. Moreover, it is sufficient to analyze the variations $\delta u(\tau)$ which are nonzero only in vicinities points where $\tilde{r}(\tau)=0$. To show this consider the McShane variation in the arbitrary interior point A_8 of the bang arc (see Fig. 10). Consider the piece $A_7 A_8 A_9 A_{10}$ of the varied trajectory $r'(\tau)$. According to Proposition 3 (see eq. (B)) the path $B_5 B_6 A_9$ is more time-efficient than $B_5 A_8 A_9$ if the varied segment $A_8 A_9$ is sufficiently small. Thus, the trajectory $A_7 B_5 B_6 A_{10}$ is more time-efficient than original segment $A_7 A_8 A_9 A_{10}$. By repeated application of the same reasoning to the modified pieces of trajectory one can replace the control $u'(\tau)$ with the more effective strategy which differs from $\tilde{u}(\tau)$ only in vicinities of the points r' with $r'_x \rightarrow 0$. Since it is sufficient to consider only this modified control policy we will rename it as $u'(\tau)$ and will refer as the initial variation in the subsequent analysis.

The characteristic piece $A_0 A_2 A_5 A_6$ of the resultant trajectory is shown in Fig. 10. Following the proof of Proposition 1 (see eq. (A2)) the path $C_0 A_2 C_1$ is less time-efficient than $C_0 A_1 B_1 C_1$. This implies that the path $A_1 B_1 C_1$ is more time-efficient than $A_1 A_2 C_1$. According to Lemma 1, the path $A_2 C_1 A_3$ is more time-efficient than the path $A_2 A_3$ associated with the free evolution. As a result, the trajectory segment $A_1 A_2 A_3$ of the $r'(\tau)$ is less time efficient than the combination of the segment $A_1 B_1 C_1$ of the trajectory $r''(\tau)$ with the segment $C_1 A_3$. By continuing the similar analysis one finally comes to conclusion that the part of trajectory $r'(\tau)$ between the points A_0 and A_5 is less time efficient than the corresponding segment of $u''(\tau)$. Applying the same reasoning to the entire trajectory $r'(\tau)$ we will reduce the original variation to the ^{0}II type control $u''(\tau)$ and trajectory $r''(\tau)$. Note that we must assume that all the singular segments where $u''(\tau)=0$ are located on the same side with respect to xz plane (otherwise the control time can be further reduced by eliminating some singular segments following the proof of Proposition 2, see eq. (B)). This mean, that all the interior bang sections of the control $u''(\tau)$ are of length $m\pi/\cos(\alpha)$ ($m \in \mathbb{N}$). Thus, the trajectory $u''(\tau)=0$ must belong to the family $\mathcal{F}[k](u''^{\text{anz}}(\tau))$ with the same index k as $\mathcal{F}[k](u''^{\text{anz}}(\tau))$ and the ansatz $u''^{\text{anz}}(\tau)$ related to $\tilde{u}^{\text{anz}}(\tau)$ via infinitesimal variation. Since $\tilde{u}^{\text{anz}}(\tau)$ is time-optimal the performances and control times associated with policies u''^{anz} and \tilde{u}^{anz} are related as $\tilde{J}^{\text{anz}} \geq J''^{\text{anz}}$ and $\tilde{T}^{\text{anz}} \leq T''^{\text{anz}}$. Consequently, $\tilde{J} \geq J''$ and $\tilde{T} \leq T''$, so that the control policies $u''(\tau)$ and $u'(\tau)$ can not be more effective than $\tilde{u}(\tau)$. The latter

conclusion completes the proof of Proposition 5.

Proof of the Lemma 1. For concreteness, consider the case $r'_y > 0$, $r_x^- > 0$. Denote $\widehat{\delta\tau} = r^+(\tau^+) - r^+(\tau^+) - \widehat{\Delta\tau}$. Using simple geometrical considerations one can find that

$$\widehat{\delta\tau}(r_x^-, r'_z) = \frac{1}{2} \sum_{s=\pm 1} \left(\arcsin \left(\frac{sr'_z - r_x^- \cot(\alpha)}{\sqrt{1-r_x^{-2}}} \right) + \frac{\arcsin \left(\frac{r_x^- \csc(\alpha) - sr'_z \cos(\alpha)}{\sqrt{1-r_z'^2 \sin^2(\alpha)}} \right)}{\sqrt{\tan^2(\alpha)+1}} \right). \quad (\text{D1})$$

By differentiating (D1) we find that $\frac{\partial}{\partial r_x^-} \widehat{\delta\tau}(r_x^-, r'_z=0) = -\frac{x^2 \sin(2\alpha) \sqrt{1-x^2 \csc^2(\alpha)}}{(x^2-1)(\cos(2\alpha)+2x^2-1)} < 0$ for any admissible $r_x^- > 0$. Similarly, one can show that $\widehat{\delta\tau}(r_x^-=0, r'_z)=0$ and $r'_z \frac{\partial}{\partial r'_z} \widehat{\delta\tau}(r_x^-, r'_z) < 0$ for any admissible $r'_z \neq 0$. Taken together, these relations lead to conclusion that $\widehat{\delta\tau}(r_x^-, r'_z) < 0$ for any admissible $r_x^- > 0$ which completes the proof for the case $r'_y > 0$, $r_x^- > 0$. Other cases can be analyzed in the same way. \square

Appendix E: Proof of Proposition 6 and 7

One can directly check that the transformation $\mathcal{S}_{\pm} = \exp(\widehat{\Delta\tau} \mathcal{L}(\pm u_{\text{max}}))$ is equivalent to the composition of rotation $\mathcal{S}_{\vec{e}_z}(\mp \xi)$ around axis \vec{e}_z by angle $\mp \xi$ with rotation $\mathcal{S}_{\vec{n}_{\pm u_{\text{max}}}}(\eta)$ around the normal vector $\vec{n}_{\pm u_{\text{max}}}$ to the plane $\lambda_{\pm u_{\text{max}}}$ by η :

$$\mathcal{S}_{\pm} = \mathcal{S}_{\vec{n}_{\pm 1}}(\eta) \mathcal{S}_{\vec{e}_z}(\mp \xi) \quad (-\pi < \eta < 0; \quad 0 < \xi < \pi), \quad (\text{E1})$$

where the domain restrictions on the values of η and ξ result from (19). Thus, the state transformation induced by any two subsequent bang arcs is equivalent to rotation around $\vec{n}_{\pm u_{\text{max}}}$ by angle 2η . This proves that the all odd (even) corner points are situated in the same plane orthogonal to $\vec{n}_{u_1^-}$ ($\vec{n}_{u_1^+}$) and parallel to \vec{e}_z . More specifically, they are located on the circles $\vec{r} \vec{n}_{\pm u_{\text{max}}} = c_0$ which are mirror images of each other in xz plane.

In order to complete proof of Proposition 6 it remains to show that $\vec{e}_z \in \lambda_{\pm u_{\text{max}}}$ (i.e. that $c_0=0$). Since it is already shown that $\vec{e}_z \parallel \lambda_{\pm u_{\text{max}}}$ it is enough to prove that there exist an least one common point with axis \vec{e}_z . Consider the infinitesimal variations $\delta\tau_i^-$ and $\delta\tau_i^+$ of the durations $\tilde{\Delta\tau}_i$ and $\tilde{\Delta\tau}_{i+1}$ of the bang arcs adjacent to arbitrary corner point $\tilde{r}_i = \tilde{r}(\tilde{\tau}_i)$, such that the transformation $\mathcal{S} = \exp(\delta\tau_i^- \mathcal{L}(\tilde{u}_i^-)) \exp(\delta\tau_i^+ \mathcal{L}(\tilde{u}_i^+))$ moves the point \tilde{r}_i into $r'_i \in \lambda_{\tilde{u}_i^-}$. In other words, we require that \tilde{r}_i and r'_i should relate by infinitesimal rotation $\mathcal{S}_{\vec{n}_{\tilde{u}_i^-}}(\delta\gamma_i)$. For convenience, we will call such variations as "sliding" ones. The form of decomposition (E1) indicates that the sliding variation at r_i shifts the locations of all subsequent corner points $\tilde{r}_{j>i} \rightarrow r'_j$ by similar rotations $\mathcal{S}_{\vec{n}_{u_j^-}}(\delta\gamma_i)$ around

the associated axes $\vec{n}_{u_j^-}$. Consider the arbitrary composition of the sliding variations, such that the trajectory start and end points remain fixed, i.e. $\sum_i \delta\gamma_i = 0$. If the extremal \tilde{u} is locally optimal then such variations should not allow the reduction of the control time T : $\sum_i \delta\tau_i \leq 0$, where $\delta\tau_i = \delta\tau_i^- + \delta\tau_i^+$. This requirement leads to the following first-order (in $\delta\tau_i$) necessary optimality condition:

$$\forall i, j : \frac{d\delta\gamma_i}{d\delta\tau_i} = \frac{d\delta\gamma_j}{d\delta\tau_j} \quad (\text{E2})$$

Using simple geometrical analysis it is possible to explicitly calculate the derivatives in (E2):

$$\frac{d\delta\gamma_i}{d\delta\tau_i} = \frac{2\sqrt{\cos^2\left(\frac{\theta}{2}\right) + u_{\max}^2}}{\frac{\tilde{r}_{i,x}}{\tilde{r}_{i,y}}\tilde{u}_i^- \sin\left(\frac{\theta}{2}\right) - \sqrt{1+u_{\max}^2} \cos\left(\frac{\theta}{2}\right)}. \quad (\text{E3})$$

We can conclude that equalities (E2) are equivalent to condition: $\frac{\tilde{r}_{i,x}}{\tilde{r}_{i,y}}\tilde{u}_i^- = \text{const}$ which directly leads to conclusion that $\tilde{\epsilon}_z \in \lambda_{\pm 1}$ and completes the proof of Proposition 6.

Remark 1. It is worth stressing that the above proof of Proposition 6 does not explicitly depend on the time optimality of the trajectory $\tilde{u}(\tau)$. Thus, its statement is generally valid for any type I extremal locally optimal with respect to small variations of control $\tilde{u}(\tau)$, including the case of fixed control time T .

The proof of Proposition 7 follows from the analysis of the higher-order terms in sliding variation along the extremal trajectory. Calculations result in the following expression:

$$\delta\gamma_i = 2\cos\left(\frac{\xi}{2}\right)\delta\tau_i - \left| \frac{\sin^3\left(\frac{\xi}{2}\right)}{u_{\max}} \right| q_i \delta\tau_i^2 + q_i^{(3)} \delta\tau_i^3 + o(\delta\tau_i^3), \quad (\text{E4})$$

where

$$q_i^{(3)} = \frac{1}{3}u_{\max}^2 \cos\left(\frac{\xi}{2}\right) \left[2\sec^2\left(\frac{\eta}{2}\right) - 3q_i^2 \tan^4\left(\frac{\eta}{2}\right) - 6\cot(\gamma_i) \left(\tan\left(\frac{\eta}{2}\right) + (q_i+1)\tan^3\left(\frac{\eta}{2}\right) \right) \right] \quad (\text{E5})$$

The necessary condition of the local optimality is thus the inequality $\sum_{i=1}^n q_i \delta\tau_i^2 \geq 0$ in which the variations $\delta\tau_i$ are subject to constraint $\sum_{i=1}^n \delta\tau_i = 0$. The power of sliding variation is in the fact that the quadratic form in the left-hand side of this inequality is diagonal (i.e. the contributions of the sliding variations $\delta\gamma_i$ are independent up to the second order in $\delta\tau_i$). Thus, optimality implies non-negativity of the following simple quadratic form:

$$Q_{kj} = \delta_{kj} q_k + q_n \quad (k, j = 1, \dots, n-1), \quad (\text{E6})$$

which can be easily rewritten in the form of statement of Proposition 7.

Appendix F: Proof of Proposition 9

Let $q' = q_{i'} < 0$ be the smallest term in the set $\{q_i\}$. By applying Proposition 7 to the corner points adjacent to i' -th we have: $q_{i'+1} + q_{i'} < 0$. These inequalities can be rewritten after some algebra as:

$$\begin{aligned} \delta\gamma_{i'} &> -\frac{\eta}{2} - \arccos\left(\sqrt{\sin^2\left(\frac{\eta}{2}\right)(\cos(\eta) + 2)}\right); \\ \delta\gamma_{i'} &< \frac{\eta}{2} + \cos^{-1}\left(\sqrt{\sin^2\left(\frac{\eta}{2}\right)(\cos(\eta) + 2)}\right) \end{aligned} \quad (\text{F1})$$

where $\delta\gamma_{i'} = (\gamma_{i'} \bmod \pi) - \frac{\pi}{2}$ ($|\delta\gamma_{i'}| < \frac{\pi+\eta}{2}$). One can show that at least one of the inequalities (F1) holds if $|\eta| < \arccos(\sqrt{2}-1)$. From the definition of η it follows that the latter inequality holds for any $u_{\max} > \sqrt{1+\sqrt{2}}$. This means that for this range of controls the i' -th corner point can be only either the left-most or the right-most corner point of time-optimal extremal. Using Fig. 5 one can accordingly improve the estimate for n_{\max} : $n_{\max} \leq \left\lceil \frac{|\eta|}{\pi-|\eta|} + 2 \right\rceil \leq 2$ for $u_{\max} > \sqrt{1+\sqrt{2}}$ Q.E.D.

Appendix G: Proof of Proposition 10

Suppose that $\tilde{r}_{i'}$ is interior corner point of the globally time-optimal solution. From (23) it follows that $\tilde{r}_{i,x} = \frac{|\tilde{u}_i^+|}{\tilde{u}_i^+} \sin(\zeta_i) \sin(\frac{\xi}{2}) \propto c \sin(\zeta_i)$, where c is some real constant. Since $|\sin(\zeta_{i'})| < \sin(\frac{\pi+\eta}{2})$ and $|\zeta_{i'} - \zeta_{i'\pm 1}| = \frac{\pi+\eta}{2}$ the following inequality holds

$$\frac{\tilde{r}_{i',x} - \tilde{r}_{i'\pm 1,x}}{\tilde{r}_{i',x}} > 0. \quad (\text{G1})$$

Proposition 3 states that the trajectory curve in vicinity of $\tilde{r}_{i',x}$ should be λ -shaped (V-shaped) in the case of $\tilde{r}_{i',x} < 0$ ($\tilde{r}_{i',x} > 0$), as shown in Fig 11. Together with (G1) this means that both left and right adjacent arcs intersect the plane $x = \tilde{r}_{i',x}$ twice and have the second common point $\{\tilde{r}_{i',x}, -\tilde{r}_{i',y}, \tilde{r}_{i',z}\}$. However, the globally time optimal trajectories can not have intersections with themselves. This contradiction proves the statement of Proposition. The associated maximal number of switchings can be directly deduced using Fig. 5.

Appendix H: Proof of Proposition 11

The statement of Proposition will be proven by contradiction. Suppose that the first of inequalities (24) is violated (the case of violation of the second inequality can be treated similarly), i.e. $\exists i : (\forall j : \tilde{r}_{i,x} \leq \tilde{r}_{j,x} \wedge \tilde{r}_{i,x} < 0)$. Using Proposition 3 we conclude that $\tilde{r}_{i,x} \leq \tilde{r}_{i-1,x}, \tilde{r}_{i+1,x}$ and that the trajectory around \tilde{r}_i is Λ -shaped: $\exists \epsilon, \forall \delta\tau \in (-\epsilon, \epsilon) : \tilde{r}_x(\tilde{r}_i + \delta\tau) < \tilde{r}_x(\tau_i)$. Similarly to the proof of Proposition 10, these observations mean

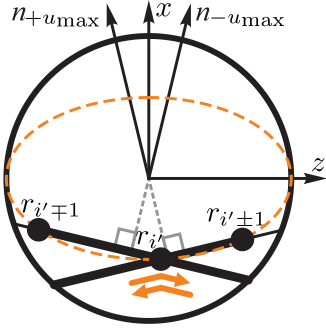


FIG. 11. Projection of the extremal in vicinity of the corner point $\tilde{r}_{i'}$ on xz -plane in the case $\tilde{r}_{i',x} < 0$. Orange dashed ellipse is the projection of intersection of the Bloch sphere with the planes $\lambda_{\pm 1}$. Arrows indicates the admissible routes of passing the point $\tilde{r}_{i'}$ according to Proposition 3.

that the both arcs $\tau \in (\tilde{\tau}_{i-1}, \tilde{\tau}_i)$ and $\tau \in (\tilde{\tau}_i, \tilde{\tau}_{i+1})$ should cross the plane $x = \tilde{r}_{i,x}$ twice and thus have the common point $\{\tilde{r}_{i,x}, -\tilde{r}_{i,y}, \tilde{r}_{i,z}\}$. However, the latter contradicts with the assumed global time optimality of the trajectory $r(\tau)$.

Appendix I: Proof of Proposition 12

Similarly to \tilde{r}^+ and \tilde{r}^- , let us introduce the new notations $r_{\pm} = \frac{\tilde{r}_1 + \tilde{r}_n}{2} \pm \text{sign}(|\tilde{r}_{1,x}| - |\tilde{r}_{n,x}|) \frac{\tilde{r}_1 - \tilde{r}_n}{2}$ for the first and the last corner points \tilde{r}_1 and \tilde{r}_n of trajectory $\tilde{r}(\tau)$, so that $|\tilde{r}_{+,x}| \geq |\tilde{r}_{-,x}|$. Using Fig. 5 we find that:

$$n_I = \left| \frac{\zeta_+ - \zeta_-}{\pi + \eta} \right| + 1 = \left| \frac{\arcsin(\tilde{r}_{+,x}\phi) - \arcsin(\tilde{r}_{-,x}\phi)}{2 \arctan(u_{\max}\phi)} \right| + 1, \quad (\text{I1})$$

where $\phi = \frac{1}{\sin(\frac{\xi}{2})}$. Eq. (I1) can be rewritten as:

$$n_I = \frac{\int_0^\phi \left| \frac{\tilde{r}_{+,x}}{\sqrt{1-\phi^2\tilde{r}_{+,x}^2}} - \frac{\tilde{r}_{-,x}}{\sqrt{1-\phi^2\tilde{r}_{-,x}^2}} \right| d\phi}{\int_0^\phi \left(\frac{u_{\max}}{1+u_{\max}^2\phi^2} \right) d\phi} + 1. \quad (\text{I2})$$

The integrands in the numerator and denominator of (I2) are monotonically increasing and decreasing functions of

ϕ in the range of interest. Since $\sin(\frac{\xi}{2}) \geq |\tilde{r}_{+,x}|$ one obtains the upper estimate $n_I \leq n_{I,\max}$, where:

$$n_{I,\max} = n|_{\phi = \frac{1}{|\tilde{r}_{+,x}|}} = \frac{\arccos(\frac{\tilde{r}_{-,x}}{\tilde{r}_{+,x}})}{2 \arctan(\frac{u_{\max}}{|\tilde{r}_{+,x}|})} + 1. \quad (\text{I3})$$

In order to make this result constructive we need estimates for $r_{\pm,x}$. Elementary analysis shows that $n_{I,\max}(\tilde{r}_{+,x}, \tilde{r}_{-,x})$ is a monotonic function of $\tilde{r}_{-,x}$ and reaches a maximum when $\text{sign}(\tilde{r}_{+,x})\tilde{r}_{-,x}$ is minimal. At the same time, $n_{I,\max}(\tilde{r}_{+,x}, \tilde{r}_{-,x})$ is a concave function of $\tilde{r}_{+,x}$ when $\tilde{r}_{+,x}\tilde{r}_{-,x} < 0$ and monotonically increasing function of $|\tilde{r}_{+,x}|$ in the range $\tilde{r}_{+,x}\tilde{r}_{-,x} > 0$. Thus, the upper estimate for $n_{I,\max}$ can be calculated by substituting into (I3) appropriate upper and/or lower estimates for $\tilde{r}_{+,x}$ and $\tilde{r}_{-,x}$. Specifically, according to Proposition 11 $|\tilde{r}_{-,x}| < |\tilde{r}_{+,x}| < |\tilde{r}_x^+|$, and $0 < |\tilde{r}_{-,x}| < |\tilde{r}_x^-|$. Substitution of these estimates results in (25a) for the case $\tilde{r}_x^+\tilde{r}_x^- < 0$ and the second of the estimates (25b) for the case $\tilde{r}_x^+\tilde{r}_x^- > 0$.

Note that the latter estimate directly accounts for the location of only one trajectory endpoint and can be further refined. Namely, due to (24) the corner points in the case $\tilde{r}_{+,x}\tilde{r}_{-,x} > 0$ are located in the range $\tilde{r}_{i,x} \in [0, \tilde{r}_x^+]$. Since the x -coordinates of the corner points are monotonic functions of the index i (see Proposition 10 and Fig. 5), the trajectory can be split into two continuous parts R_1 and R_2 such that all $n_{R_1}(n_{R_2})$ corner points in the segment $R_1(R_2)$ belong to the range $\tilde{r}_{i,x} \in (\tilde{r}_x^-, \tilde{r}_x^+]$ ($\tilde{r}_{i,x} \in [0, \tilde{r}_x^-]$), and their junction point \tilde{r}_c is chosen such that $\tilde{r}_{c,x} = \tilde{r}_x^-$. Using these range estimates and the extremal properties of function (I3) we obtain

that $n_{R_1} \leq \frac{\arccos(\frac{\tilde{r}_x^-}{\tilde{r}_x^+})}{2 \arctan(\frac{u_{\max}}{\tilde{r}_x^+})} + 1$. Let us show that $n_{R_2} \leq 3$ (which will prove the first estimate in (25b)). Indeed, the duration $\tilde{\Delta}\tau_{R_2}$ of this segment can not exceed π (the maximal duration of the trajectory with $\tilde{u}(\tau) = 0$ connecting \tilde{r}^- and \tilde{r}_c). At the same time, according to eq. (19) the minimal duration of each arc of the bang-bang trajectory is $\frac{\pi}{2} \cos \alpha$. Thus, the number of the interior bang segments of duration $\tilde{\Delta}\tau$ in the case $u_{\max} \leq 1$ can not exceed $[2\sqrt{2}] = 2$, i.e. $n_{R_2} \leq 3$ (the same restriction for the case $u_{\max} > 1$ trivially follows from Proposition (8)). Hence, Proposition is completely proven.

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